Pergamon

Computers Math. Applic. Vol. 33, No. 4, pp. 49–51, 1997

0898-1221/97 $17.00 + 0.00

PII: S0898-1221(97)00006-0

Uniqueness of Solution for Mixed Problems Related to the Generalized Wave Equation

L. JÓDAR AND P. ALMENAR
Departamento de Matemática Aplicada
Universidad Politécnica de Valencia,
P.O. Box 22012, 46071 Valencia, Spain
ljodar@pleiades.upv.es

(Received and accepted August 1996)

Abstract—In this note, the uniqueness of solution for mixed problems recently studied in [1] is proved. This fact guarantees that the numerical solutions constructed in [1] converge to the unique solution of the problem.

Keywords—Generalized wave equation, Uniqueness.

1. INTRODUCTION

In the recent paper [1], the authors proposed a method for constructing continuous-numerical solutions with prefixed accuracy, of the mixed problem:

$$c(t)u_{xx} = u_{tt}, \qquad t > 0, \quad 0 < x < p, \tag{1}$$

$$u(0,t) = u(p,t) = 0, \qquad t > 0, \tag{2}$$

$$u(x,0) = f(x), \qquad 0 \le x \le p, \tag{3}$$

$$u_t(x,0) = g(x), \qquad 0 \le x \le p, \tag{4}$$

where

$$c(t) \text{ is a two times continuously differentiable function in } [0,+\infty[\text{ with } c(t) > 0 \text{ and } |c'(t)| + |c^{(2)}(t)| > 0, \text{ for all } t \ge 0, \tag{5}$$

$$f(x) \text{ is a four times differentiable function in } [0,p],\ f^{(4)}(x) \text{ is piecewise continuous and } f(0) = f(p) = f^{(2)}(0) = f^{(2)}(p) = 0, \tag{6}$$

$$g(x) \text{ is a three times differentiable function in } [0,p],\ g^{(3)}(x) \text{ is piecewise continuous and } g(0) = g(p) = g^{(2)}(0) = g^{(2)}(p) = 0. \tag{7}$$

However, without the uniqueness guarantee, different numerical solutions could approximate to different solutions of the problem, and the interest of the proposed numerical method would be very limited. This motivates the study of the uniqueness.

This work has been supported by the Generalitat Valenciana Grant GV1118/93 and the D.G.I.C.Y.T. Grant PB93-381.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

2. THE UNIQUENESS

We'll begin this section with a lemma that will play a crucial role in the study of the uniqueness.

LEMMA 1. *Let $T(x,t)$ be a solution of problem (1)–(5) with $f = g = 0$, and let us suppose that:*

$$T(x,t_0) = T_t(x,t_0) = 0, \qquad 0 \le x \le p, \quad t_0 \ge 0, \tag{8}$$

$$c'(t) \text{ has a constant sign in }]t_0,t_1[. \tag{9}$$

Then $T(x,t) = 0$, for $(x,t) \in [0,p] \times [t_0,t_1[$.

PROOF. We distinguish three cases.

CASE 1: $c'(t) < 0$ for $t \in]t_0,t_1[$. Let $T(x,t)$ be a solution of problem (1)–(5) with $f = g = 0$, satisfying (8) and (9), and let $E(t)$ be defined by

$$E(t) = \frac{1}{2}\int_0^p \left[c(t)T_x^2(x,t) + T_t^2(x,t)\right] dx, \qquad t_0 \le t \le t_1. \tag{10}$$

Taking derivatives in (10), from Leibniz's rule, the hypotheses, and integrating by parts, it follows that

$$\begin{aligned} E'(t) &= \frac{1}{2}\int_0^p c'(t)T_x^2\,dx + c(t)\int_0^p T_xT_{xt}\,dx + \int_0^p T_tT_{tt}\,dx \\ &= \frac{1}{2}\int_0^p c'(t)T_x^2\,dx + c(t)\left[T_xT_t\right]_0^p - c(t)\int_0^p T_tT_{xx}\,dx + \int_0^p T_tT_{tt}\,dx \\ &= \frac{1}{2}\int_0^p c'(t)T_x^2\,dx + c(t)\left[T_x(p,t)T_t(p,t) - T_x(0,t)T_t(0,t)\right]. \end{aligned}$$

Hence,

$$E'(t) = \frac{1}{2}\int_0^p \left[c'(t)T_x^2\right] dx \le 0, \qquad t \in [t_0,t_1[, \tag{11}$$

because $T_x(x,t_0) = 0$ and $c'(t) < 0$ in $]t_0,t_1[$. Note that by (8), one gets $E(t_0) = 0$ and by definition $E(t) \ge 0$. By (11), one concludes that $E(t) = 0$ for $t \in [t_0,t_1[$. By continuity of $T_x^2(\cdot,t)$ and $T_t^2(\cdot,t)$, and from (10), it follows that

$$T_x(x,t) = T_t(x,t) = 0, \qquad \text{for } 0 \le x \le p, \quad t_0 \le t < t_1. \tag{12}$$

By (8) and (12) one gets $T(x,t) = T(x,t_0) = 0$, for $(x,t) \in [0,p] \times [t_0,t_1[$.

CASE 2: $c'(t) > 0$ for $]t_0,t_1[$. Given $T(x,t)$, let $H(t)$ be the function defined by

$$H(t) = \frac{1}{2}\int_0^p \left[T_x^2(x,t) + \frac{T_t^2(x,t)}{c(t)}\right] dx, \qquad t_0 \le t \le t_1. \tag{13}$$

From the hypotheses, Leibniz's rule, and the method of integration by parts, the derivative of $H(t)$ can be written in the form

$$\begin{aligned} H'(t) &= \int_0^p (T_xT_{xt})\,dx + \frac{1}{c(t)}\int_0^p (T_tT_{tt})\,dx - \frac{c'(t)}{2c^2(t)}\int_0^p T_t^2\,dx \\ &= \left[T_xT_t\right]_0^p - \int_0^p (T_{xx}T_t)\,dx + \int_0^p (T_{xx}T_t)\,dx - \frac{c'(t)}{2c^2(t)}\int_0^p T_t^2\,dx, \\ H'(t) &= -\frac{c'(t)}{2c^2(t)}\int_0^p T_t^2\,dx \le 0, \qquad t_0 \le t < t_1, \end{aligned} \tag{14}$$

because $T_t(x,t_0) = 0$ and $c'(t) > 0$, for $t_0 < t < t_1$. By (8) and (13) it follows that $H(t_0) = 0$ and $H(t) \ge 0$, for $t_0 < t < t_1$. Since by (14), the function $H(t)$ is decreasing in $]t_0,t_1[$, one gets that $H(t) = H(t_0) = 0$, for $t_0 \le t < t_1$.

By continuity of $T_x^2(\cdot,t)$ and $T_t^2(\cdot,t)$, and by (13), one concludes that $T_x(x,t) = T_t(x,t) = 0$, for $0 \le x \le p$, $t_0 \le t < t_1$. Hence, $T(x,t) = 0$ for $(x,t) \in [0,p] \times [t_0,t_1[$.

CASE 3: $c'(t) = 0$ for $]t_0,t_1[$. In this case, $c(t)$ is a constant function in $[t_0,t_1]$. Given $T(x,t)$, let us consider $E(t)$ defined as in (10) with $c(t) = c > 0$,

$$E(t) = \frac{1}{2}\int_0^p \left[c(t)T_x^2(x,t) + T_t^2(x,t)\right] dx, \qquad t_0 \le t \le t_1. \tag{15}$$

As $T(x,t)$ is a solution of (1)–(5) with $f = g = 0$, using the argument of Case 1, it is easy to show that $E'(t) = 0$ for $t_0 \le t < t_1$. As $E(t_0) = 0$, one gets $E(t) = E(t_0) = 0$, $t_0 \le t < t_1$. By continuity of $T_x^2(\cdot,t)$ and $T_t^2(\cdot,t)$ and by (15), it follows that $T_x(x,t) = T_t(x,t) = 0$ for $(x,t) \in [0,p] \times [t_0,t_1[$. Hence, $T(x,t) = 0$ for $(x,t) \in [0,p] \times [t_0,t_1[$.

REMARK 1. Under the hypotheses (8) the sign of the function $c'(t)$ can only change a finite number of times within a bounded interval $[0,p]$. In fact, in other case, there would exist an infinite sequence $\{t_k\}_{k=1}^{\infty}$ in $[0,T]$ with $t_k \ne t_{k'}$ for $k \ne k'$, such that

$$\operatorname{sign} c'(t_k) = -\operatorname{sign} c'(t_{k+1}).$$

By the mean value theorem there exists a sequence $\{s_k\}_{k=1}^{\infty}$ with $t_k < s_k < t_{k+1}$, such that $c'(s_k) = 0$ for $k \ge 1$. By Bolzano-Weierstrass's Theorem, there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} s_{n_k} = s^* \in [0,T]$, and

$$0 = \lim_{k\to\infty} c'(s_{n_k}) = c'(s^*). \tag{16}$$

Again by the mean value theorem, there exists a sequence of points $\xi_k \in]s_{n_k}, s_{n_{k+1}}[$ such that $c''(\xi_k) = 0$ for $k \ge 1$. Furthermore, $\lim_{k\to\infty} \xi_k = s^*$ and

$$c''(s^*) = \lim_{k\to\infty} c''(\xi_k) = 0. \tag{17}$$

But (16) and (17) contradict (8). Thus, the sign of $c'(t)$ can only change a finite number of times in $[0,T]$.

THEOREM 1. *Under the hypotheses (5)–(7), the problem (1)–(4) only admits one solution.*

PROOF. The existence of solution was proved in [1]. To prove the uniqueness, it is sufficient to prove the uniqueness in any bounded domain $D = [0,p] \times [0,T]$, where T is any fixed positive number. Also, by linearity of problem (1)–(4), it is sufficient to prove that the unique solution of problem (1)–(4), with $f = g = 0$, is the zero solution $T(x,t) = 0$ for $(x,t) \in [0,p] \times [0,T]$. If the sign of $c'(t)$ does not change in $[0,T]$, then the result is proved in Lemma 1. If not, let us suppose that there exists a partition $0 = t_0 < t_1 < \cdots < t_N = T$, so that $c'(t)$ has a constant sign in $]t_i,t_{i+1}[$ for $0 \le i \le N-1$. By application of Lemma 1 to the interval $[0,t_1[$, it follows that $T(x,t) = 0$ for $(x,t) \in [0,p] \times [0,t_1[$, and by continuity one gets $T(x,t) = 0$ and $T_t(x,t) = 0$ for $(x,t) \in [0,p] \times [0,t_1]$.

Inductively by application of Lemma 1, one gets $T(x,t) = 0$ and $T_t(x,t) = 0$ for $(x,t) \in [0,p] \times [t_1,t_2]$, and finally $T(x,t) = 0$ for $(x,t) \in [0,p] \times [0,T]$. Hence, the result is established.

REFERENCES

1. L. Jódar and P. Almenar, Accurate continuous numerical solutions of time dependent mixed partial differential problems, *Computers Math. Applic.* **32** (2), 5–19, (1996).